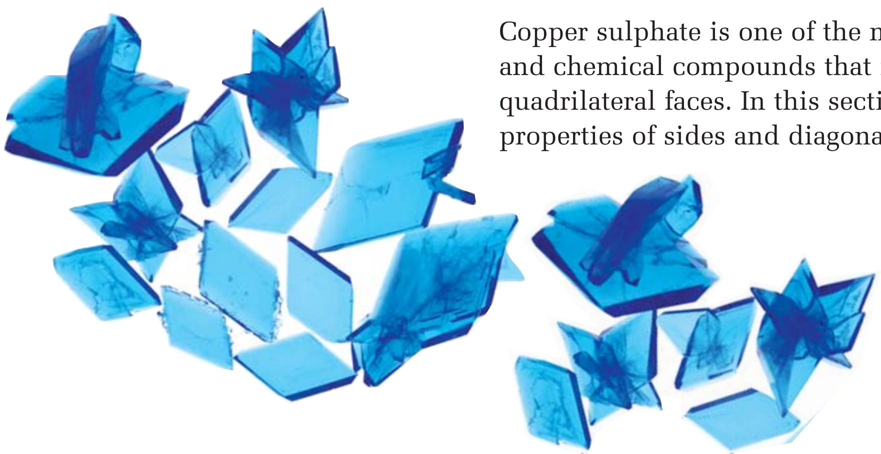
7.5 Midpoints and Diagonals in Quadrilaterals



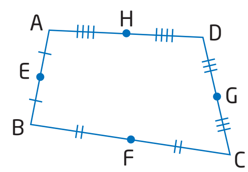
Copper sulphate is one of the many minerals and chemical compounds that form crystals with quadrilateral faces. In this section, you will examine the properties of sides and diagonals of quadrilaterals.

Investigate

What are the properties of the midpoints of the sides of a quadrilateral?

Tools
- ruler
- protractor

Method 1: Use Paper and Pencil



1. Draw a large quadrilateral ABCD. Measure the four sides and mark the midpoints, with E the midpoint of AB, F the midpoint of BC, and so on.
2. Draw line segments joining E to F, F to G, G to H, and H to E. What type of quadrilateral does EFGH appear to be?
3. Measure and compare the line segments in the smaller quadrilateral. What relationships are there among the lengths of these line segments?
4. Measure the interior angles of quadrilateral EFGH with a protractor. Mark these measures on your drawing.
5. If the co-interior angles formed by a transversal and two line segments are supplementary, the two segments are parallel. Are any of the sides of the quadrilateral EFGH parallel?
6. **Reflect** Do your answers to steps 3 to 5 confirm your conjecture in step 2? Explain.
7. Compare your results with those of your classmates.

8. **Reflect** Do you think that joining the midpoints of the sides of any quadrilateral always produces the same type of geometric shape? Explain your reasoning.

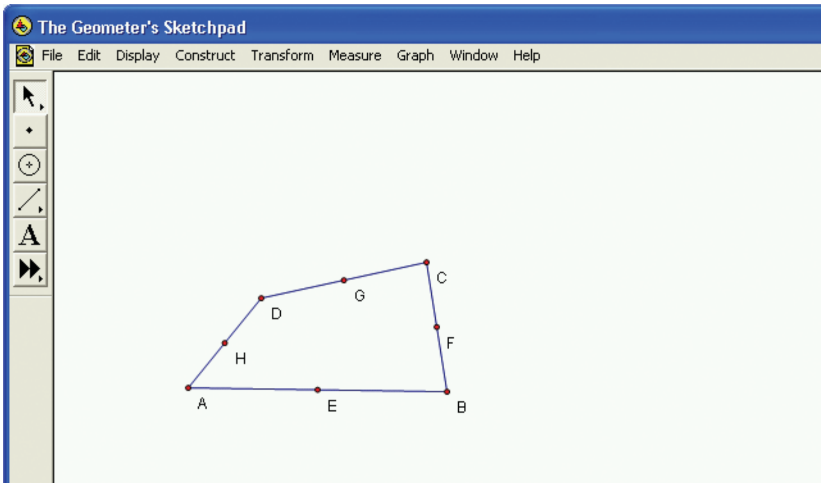
Tools

- computer with *The Geometer's Sketchpad®*

Method 2: Use *The Geometer's Sketchpad®*

1. Turn on automatic labelling of points. From the **Edit** menu, choose **Preferences**. Click on the **Text** tab, check **For All New Points**, and click on **OK**.

2. Construct any quadrilateral ABCD. Construct the midpoints of the four sides by selecting the sides and then choosing **Midpoints** from the **Construct** menu.



3. Construct line segments EF, FG, GH, and HE. What type of quadrilateral does EFGH appear to be?

4. Measure and compare the sides of the smaller quadrilateral. What relationships are there among these lengths?

5. Do any of these relationships change if you drag any of the vertices of ABCD to a different location?

6. Measure all the interior angles of quadrilateral EFGH.

7. If the co-interior angles formed by a transversal and two line segments are supplementary, the two segments are parallel. Calculate the sums of adjacent interior angles to see if any of the sides of quadrilateral EFGH are parallel. Does moving a vertex of the original quadrilateral ABCD change any of the angle sums?

8. **Reflect** Do your measurements confirm your conjecture in step 3? Do you think that joining the midpoints of the sides of any quadrilateral produces the same type of geometric shape? Explain your reasoning.

Technology Tip

You can also do this investigation using Cabri Jr. on a graphing calculator. For step-by-step instructions, follow the links at www.mcgrawhill.ca/links/principles9.

Example 1 Diagonals of a Parallelogram

Show that the diagonals of a parallelogram bisect each other.

Solution

Use *The Geometer's Sketchpad*®. Turn on automatic labelling of points.

Construct line segment AB and point C above it. Connect B to C with a line segment.

Select point C and line segment AB. Choose **Parallel Line** from the **Construct** menu.

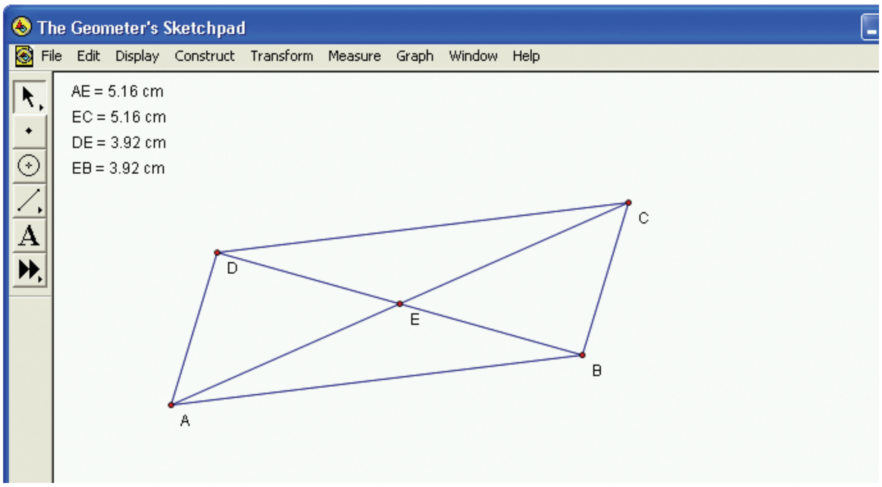
Select point A and line segment BC. Then, choose **Parallel Line** from the **Construct** menu again.

Select the two lines that you constructed. Then, choose **Intersection** from the **Construct** menu.

Select the two lines again, and choose **Hide Parallel Lines** from the **Display** menu.

Construct line segments from C to D and from D to A. Then, construct diagonals AC and BD. Select the two diagonals and choose **Intersection** from the **Construct** menu.

Measure each line segment from point E to a vertex. These measurements show that EB = DE and AE = CE. The diagonals of this parallelogram bisect each other.



Now, drag each of the vertices to various new locations. ABCD remains a parallelogram, and the lengths of EB and DE remain equal, as do the lengths of AE and CE. Therefore, the diagonals of any parallelogram bisect each other.

Tools
- computer with *The Geometer's Sketchpad*®

Technology Tip

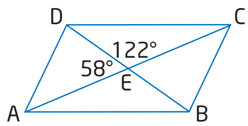
The keyboard shortcut for the Hide option is Ctrl+H.

Example 2 Use a Counter-Example

Jody conjectured that the diagonals of a parallelogram are always perpendicular to each other. Is she correct?

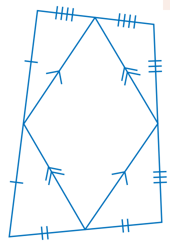
Solution

Draw any parallelogram ABCD that is not a rhombus. Draw the diagonals BD and AC. Then, measure the angles at the point E where the diagonals intersect. None of these angles is a right angle. Since perpendicular lines meet at right angles, the diagonals in this parallelogram are not perpendicular to each other. Therefore, the conjecture that the diagonals of a parallelogram are always perpendicular to each other is incorrect.

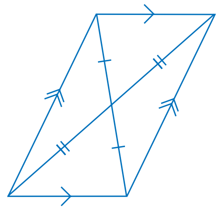


Key Concepts

- Joining the midpoints of the sides of any quadrilateral produces a parallelogram.



- The diagonals of a parallelogram bisect each other.



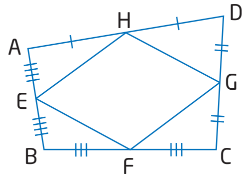
Communicate Your Understanding

C1 Describe how you can tell if two sides of a quadrilateral are parallel.

C2 Describe how you could fold a diagram of a parallelogram to show that its diagonals bisect each other.

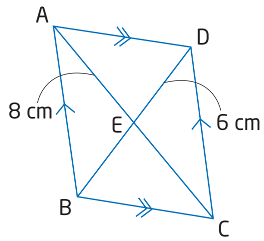
Practise

1. Which line segments in this diagram are parallel?

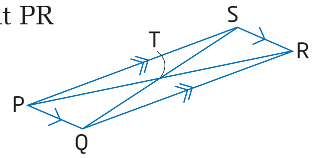


For help with questions 2 and 3, see Example 1.

2. Calculate the lengths of BE, CE, AC, and BD.

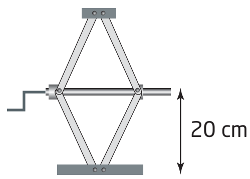


3. Calculate the lengths of PT and ST given that PR measures 14 m and QS measures 10 m.



Connect and Apply

4. Drivers often use a scissor jack when changing a tire. The crank turns a threaded shaft that pulls the sides of the hinged parallelogram toward each other, raising the top of the jack. How high will the top of the jack be when the shaft is 20 cm from the base?

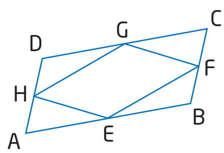


Did You Know?

Multistage scissor mechanisms can raise a platform 15 m or more. These machines are often used in construction and movie-making.

5. Construct a parallelogram, a rectangle, a rhombus, and a square. Draw the diagonals for each quadrilateral. Then, use your drawings to determine in which of the four quadrilaterals the diagonals
   a) bisect each other
   b) have the same length
   c) intersect at 90°
   d) bisect each other at 90°

6. Construct a parallelogram ABCD. Let E be the midpoint of AB, F be the midpoint of BC, G be the midpoint of CD, and H be the midpoint of DA. Connect EF, FG, GH, and HE to form a new parallelogram EFGH. Under what conditions is EFGH a rhombus?



7. For each statement, either explain why it is true or draw a counter-example to show that it is false.
   a) Any diagonal of a quadrilateral bisects its area.
   b) Any line segment joining the midpoints of opposite sides of a parallelogram bisects its area.

8. On grid paper, draw a square PQRS and mark the midpoints of the four sides. Label these midpoints W, X, Y, and Z.

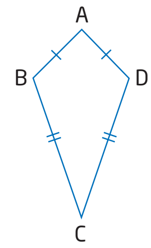
a) What type of quadrilateral is WXYZ?

b) How is the area of WXYZ related to the area of PQRS? Explain your reasoning.

c) What shape will WXYZ become if PQRS is stretched to form a rectangle? Support your answer with a drawing.

d) Will the relationship between the areas of WXYZ and PQRS change when PQRS is stretched into a rectangle? Explain.

9. a) Draw a quadrilateral ABCD with AB = AD and BC = DC.



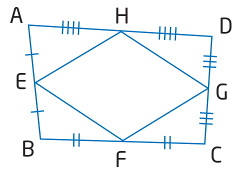
b) At what angle do the diagonals of the quadrilateral intersect?

c) Join the midpoints of the sides of the quadrilateral to form a smaller quadrilateral EFGH. What type of quadrilateral is EFGH?

d) Make a conjecture about how the area of EFGH is related to the area of ABCD.

e) Describe how you can use geometry software to test your conjecture.

10. In this diagram, line segments joining the midpoints of the four sides of a quadrilateral form a smaller quadrilateral inside the original quadrilateral.



a) How do you think the area of the smaller quadrilateral compares to the area of the original quadrilateral?

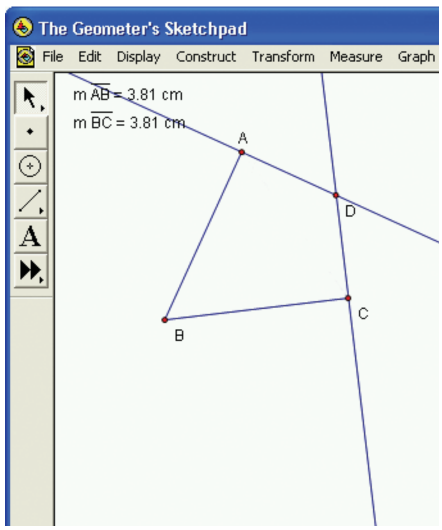
b) Describe how you could confirm your conjecture.

11. Use Technology

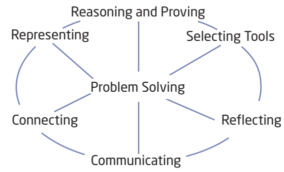
a) Construct ∠ABC with BA = BC. Construct a line perpendicular to AB at point A and a line perpendicular to BC at point C. Label the intersection of these lines D.

b) Show that AD = CD.

c) Show that BD bisects ∠ABC.



Achievement Check



12. While reviewing for a geometry test, two of your friends find that they have different answers for several true/false questions. The two friends ask you to help them decide which answers are right. For each of these statements, either use a diagram to help explain why the statement is true, or draw a counter-example and explain why the statement is false.

a) A line segment joining the midpoints of two sides of a triangle bisects the area of the triangle.

b) Any diagonal of a parallelogram bisects its area.

c) A line segment joining the midpoints of the parallel sides of a trapezoid bisects its area.

d) A line segment joining the midpoints of opposite sides of a quadrilateral bisects its area.

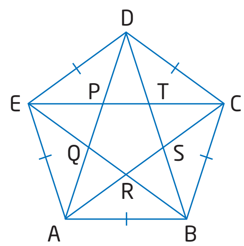
Extend

13. Use congruent triangles to show that the diagonals of a parallelogram bisect each other.

14. **a)** Draw a quadrilateral ABCD with AB = CD and AD = CB.

b) Show that this quadrilateral must be a parallelogram.

15. This diagram shows all possible diagonals for a regular pentagon ABCDE.



a) Is PQRST a regular polygon? How do you know?

b) Is PQRST similar to ABCDE? Explain.

c) Compare the lengths of the sides of PQRST to those of ABCDE.

d) Make a conjecture about how the ratio of the areas of PQRST and ABCDE is related to the ratio of their side lengths.

e) Use geometry software to test your conjecture. Describe your results.

16. **Math Contest**

a) How many line segments can be constructed between 10 points in a plane? Assume that no three points are on the same straight line.

b) Twelve people arrive at a meeting, one at a time. Each of these people shakes hands with everyone who is already there. How many handshakes have occurred once the 12th person has finished shaking hands?

17. **Math Contest**

a) Find a formula for the total number of line segments that can be constructed between n points in a plane if no three points are on the same straight line.

b) Use your answer to part a) to find a formula for the number of diagonals in a polygon with n sides.